Exercise 1. Prove: a connected graph G is Eulerian if and only if every edge of G is contained in an odd number of cycles.

Pf:

($\Rightarrow$)

Suppose G is Eulerian.

Let $e = uv$ be any edge of G.

Since G is Eulerian, G has an Euler circuit.

Delete e from this Euler circuit.

We obtain an Euler trail W from u to v in $H = G - e$.

By the path-parity lemma, the graph formed by the edges of W contains an odd number of u-v paths.

Since W uses every edge of H, there are exactly the u-v paths in H.

Actually, the u-v paths in H are in one-to-one correspondence with the cycles of G containing e.

Hence, $c(e) \equiv 1 \pmod 2$, where $c(e)$ denotes # cycles containing e.

($\Leftarrow$)

Suppose every edge of G contains an odd number of cycles.

Let v be any vertex.

Define $S_v = \sum_{e \ni v} c(e)$.

Since $c(e)$ is odd for every e in G.

$S_v \equiv \sum_{e \ni v} 1 = d_G(v) \pmod 2$.

Let's do a double counting.

Every cycle containing v uses exactly 2 edges incident with v.

Hence, S_v is even.

Thus, $S_v \equiv 0 \pmod 2$, $d_G(v) \equiv 0 \pmod 2$

By Euler's theorem, G is Eulerian.

□.

note: the path-parity lemma refers to # paths between any 2 vertices of an Eulerian Graph is even.

Though I couldn't find it in slides, I used it. The key to prove it is to construct an Involution without fixed points.

Exercise 2. Let G be a simple graph with no cut vertex and at least $n \geq 3$ vertices. Prove: if for any two vertices x and y at distance 2,

$$\max\{d_G(x),\, d_G(y)\} \geq \frac{n}{2} \qquad \text{(called the Fan condition)},$$

then G is Hamiltonian.
(see Geng-Hua Fan, *New sufficient conditions for cycles in graphs*, J. Combin. Theory Ser. B, **37** (1984), pp. 221–227.)

Pf:

Let $S = \{ v \in V(G) : d_G(v) \geq n/2 \}$.
$T = V(G) \setminus S$.

① Find a longest path with 2 end-vertices which are in S.

Among all longest path in G, we choose one with the maximum number of end-vertices which are in S.
Write it as

$$P = v_1 v_2 \cdots v_k .$$

We prove $v_1, v_k \in S$.
Suppose $v_1 \notin S$.
Since P is longest, $N_G(v_1) \subset V(P)$.
Suppose $v_1 v_j \in E(G)$, $j \geq 3$.
We can construct $P' = v_{j-1} v_{j-2} \cdots v_1 v_j v_{j+1} \cdots v_k$
which is also longest.
If $v_1 v_{j-1} \notin E(G)$, then $v_1 - v_j - v_{j-1}$ is a path of length two.

Thus $d_G(v_1, v_{j-1}) = 2$.
According to the Fan condition, $v_{j-1} \in S$.
A CONTRADICTION for the maximum number of end-vertices which are in S.
Thus $v_1 v_{j-1} \in E(G)$.
Hence, by induction, for some m,
$N_G(v_1) = \{v_2, \cdots, v_m\}$.

1. If $m = k$, then $P + v_1 v_k$ is a cycle.
If there exists $v_0 \notin V(P)$, by connectedness, we can construct a longer path, A CONTRADICTION.
Hence $V(P) = V(G)$, $P + v_1 v_k$ is hamiltonian.

2. If $m \neq k$, we choose P so that m is largest.
Since $N_G(v_1) = \{v_2, \cdots, v_m\}$,
$v_1 v_{i+1} \in E(G)$, $1 \leq i \leq m-1$.
We can construct $Q_i = v_i v_{i-1} \cdots v_1 v_{i+1} v_{i+2} \cdots v_k$,
which is also longest.
By hypothesis, v_i must be in T.
If $v_i v_s \in E(G)$, $s > m$,
then in Q_i, this contradicts the maximal choice of m.
Hence $N_G(v_i) \subset \{v_1, v_2, \cdots, v_m\}$, $1 \leq i \leq m-1$.
Then $E(\{v_1, \cdots, v_{m-1}\}, \{v_{m+1}, \cdots, v_k\}) = \phi$,
v_m is a cut point, A CONTRADICTION.
Thus $v_1 \in S$. By symmetry, $v_k \in S$.

② Construct a Hamilton cycle from P.

If $N_G(v_1) \cap N_G(v_k) = \phi$, then

$$d_G(v_1) + d_G(v_k) = |N_G(v_1) \cup N_G(v_k)|$$
$$\leq k - 1$$
$$\leq n - 1$$

A CONTRADICTION.

Thus $N_G(v_1) \cap N_G(v_k) \neq \phi$.

Choose $v_i \in N_G(v_1) \cap N_G(v_k)$,

we construct $C = v_1 v_2 \cdots v_i v_k v_{k-1} \cdots v_{i+1} v_1$,

which is a cycle of $V(P)$.

By connectedness, C must contain all vertices in G, otherwise we can construct a longer path.

Thus C is a Hamilton cycle, and G is Hamiltonian.

□.

Exercise 3. Prove the following statement (Proposition 12.5): the line graph $L(G)$ of any graph G is claw-free.

pf. Let $K_{1,3} \in \text{Ind}(L(G))$.

Suppose e be the central vertex of $K_{1,3}$, e_1, e_2, e_3 be the leaf vertices.

By pigeonhole principle, there exists one vertex in e which is incident to at least two edges in $\{e_1, e_2, e_3\}$.

Without less of generality, we assum these are e_1 and e_2.

Thus in $L(G)$, e_1 and e_2 is connected.
A CONTRADICTION.
$K_{1,3} \notin \mathrm{Ind}(L(G))$, $\forall G$. □.

Exercise 4. Let x, y, z be three distinct vertices of a strongly connected digraph D. Prove:

$$d_D(x,z) \le d_D(x,y) + d_D(y,z).$$

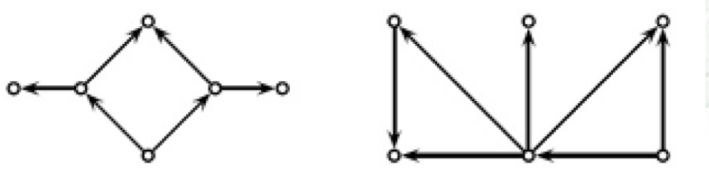
pf: Let $P^1 = (x=) v_1^1 v_2^1 \cdots v_{k_1}^1 (= y)$
be the shortest path between x and y.
$P^2 = (y=) v_1^2 v_2^2 \cdots v_{k_2}^2 (= z)$
be the shortest path between y and z.
Then $W = (x=) v_1^1 v_2^1 \cdots v_{k_1}^1 v_2^2 \cdots v_{k_2}^2 (= z)$
is a walk between x and z.
Thus $d_D(x,z) \le E(W) = E(P^1) + E(P^2)$
$= d_D(x,y) + d_D(y,z)$.
□.

Exercise 5. Let A be the adjacency matrix of a (directed or undirected) graph D, and I the $n \times n$ identity matrix. The polynomial

$$P_D(\lambda) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

is called the *characteristic polynomial* of D; the roots of $P_D(\lambda)$ are called the *characteristic roots* of D. The characteristic roots together with their multiplicities form the *spectrum*.

(a) Find the characteristic polynomials (both are of degree 6) and characteristic roots of the following two graphs:



Note: Graphs with the same spectrum as in the figure above are called *cospectral graphs*. Clearly, a graph uniquely determines its spectrum; the converse is one of the difficult open problems in graph theory.

Let D_1, D_2 be the graphs above.
A_1, A_2 be the adjacent matrices.

Since D_1, D_2 are DAG,
A_1, A_2 is nilpotent.
The length of the longest walk in D_1 is 2.
hence $A_1^2 \neq 0$, $A_1^3 = 0$. $m_{A_1}(t) \mid t^3$.
By minimality, $m_{A_1}(t) = t^3$.
The eigenvalue must be 0.
Thus $P_{D_1}(\lambda) = \lambda^6$.

The length of the longest walk in D_2 is 3.
hence $A_2^3 \neq 0$, $A_2^4 = 0$. $m_{A_2}(t) \mid t^4$.
By minimality, $m_{A_2}(t) = t^4$.
The eigenvalue must be 0.
Thus $P_{D_2}(\lambda) = \lambda^6$.