Week 3

Paths and cycles are among the most basic structures in graph theory, and questions concerning their existence form one of the main topics in this area. Depending on the setting, one may look for short cycles, paths or cycles of a prescribed length, sufficiently long paths or cycles, Hamiltonian paths or cycles,which are among the most basic spanning structures, or paths and cycles whose lengths lie in prescribed residue classes modulo an integer. Another important direction is to study the existence of cycles of several different lengths simultaneously, for example, cycles of consecutive lengths or even cycle lengths forming an arithmetic progression. A fundamental problem is therefore to determine which conditions on the number of edges, minimum degree, connectivity, or other structural properties guarantee the existence of such paths and cycles. If you are interested, you may also search for recent research papers on related topics independently.

Exercises

1. Prove: let D be a simple graph, and $k = \max\{\delta^+, \delta^-\}$, then D contains a directed path of length at least k.

Proof. Let $P = v_0v_1\cdots v_l$ be a longest directed path of length l in D. Since D is a simple graph, by the choice of P,

$$N^+(v_l) \subseteq V(P) \setminus \{v_l\}, N^-(v_1) \subseteq V(P) \setminus \{v_1\}.$$

Therefore, $l \geq \max\{d^+(v_l), d^-(v_1)\} \geq k$. □

2. Let G be a simple graph of order n, and $\omega = \omega(G)$. Prove: (a) $e(G) \leq \frac{1}{2}(n-\omega)(n-\omega+1)$
(b) If $e(G) > \frac{1}{2}(n-1)(n-2)$, then G is connected.

Proof. (a) Let the components of G be $G_1, G_2, \ldots, G_\omega$, where $1 \leq n_i = |V(G_i)| \leq n-\omega+1$ for $1 \leq i \leq \omega$. Since each G_i is a simple graph,

$$e(G_i) \leq \binom{n_i}{2}.$$

Hence

$$\begin{aligned} e(G) = \sum_{i=1}^{\omega} e(G_i) &\leq \sum_{i=1}^{\omega} \binom{n_i}{2} \\ &= \frac{1}{2}\sum_{i=1}^{\omega} n_i(n_i-1) \\ &\leq \frac{1}{2}(n-\omega+1)\sum_{i=1}^{\omega}(n_i-1) \\ &= \frac{1}{2}(n-\omega)(n-\omega+1). \end{aligned}$$

(b) Suppose, to the contrary, that G is disconnected. Then $\omega \geq 2$. By part (a),

$$e(G) \leq \frac{1}{2}(n-\omega)(n-\omega+1) \leq \frac{1}{2}(n-2)(n-1),$$

which contradicts

$$e(G) > \frac{1}{2}(n-1)(n-2).$$

Hence G is connected.

□

Remark. In graph theory, we also frequently use local modifications to obtain the desired result. In (a), we may move vertices outside the largest component into the largest component in such a way that the number of edges does not decrease. Therefore, the maximum number of edges is attained when every component other than the largest one consists of an isolated vertex.

Second proof of (a). Choose one vertex from each component of G, and let U be the set of chosen vertices, where $|U| = \omega$. Set $W = V(G) \setminus U$. Since the vertices in U lie in distinct components, $e(G[U]) = 0$. Moreover, every vertex in W is adjacent to at most one vertex in U. Hence,

$$\begin{aligned} e(G) &= e(G[W]) + e(G[U]) + e(U, W) \\ &\leq \binom{|W|}{2} + 0 + |W| \\ &\leq \binom{n-\omega}{2} + n - \omega \\ &= \frac{1}{2}(n-\omega)(n-\omega+1). \end{aligned}$$

□

3. Let G be a non-bipartite simple graph of order n, $k \geq 2$, and $\delta(G) > \left\lfloor \frac{2n}{2k+1} \right\rfloor$. Prove that G contains an odd cycle of length $\leq 2k-1$.

Proof. Since G is non-bipartite, it contains an odd cycle. Let C be a shortest odd cycle in G, and let $l = |C|$. If $l = 3$, then there is nothing to prove. Hence, we may assume that $l \geq 5$. By the choice of C, the cycle C is chordless, since any chord of C would divide C into two cycles, one of which is odd and shorter than C.

Moreover, every vertex $u \in V(G) \setminus V(C)$ has at most two neighbors on C. Indeed, if u had three neighbors on C, these three vertices would divide C into three internally disjoint paths. Since l is odd, at least one of these paths has odd length. Choose the shortest one, together with the two edges joining its endpoints to u, it would form an odd cycle shorter than C, a contradiction. Therefore,

$$\begin{aligned} \sum_{v \in V(C)} d_G(v) &= 2|E(C)| + e\big(V(C), V(G) \setminus V(C)\big) \\ &\leq 2l + 2(n-l) = 2n. \end{aligned}$$

Since $\sum_{v \in V(C)} d_G(v) \geq \delta(G) l$, we obtain $\delta(G) \leq \frac{2n}{l}$. Combining this with the hypothesis $\delta(G) > \frac{2n}{2k+1}$, we get $l < 2k+1$. As l is odd, it follows that

$$l \leq 2k-1.$$

□