# Week 10

**1.** (a) Prove that if $G$ is a simple graph on $n \geq 2$ vertices with $\delta(G) \geq \lfloor n/2 \rfloor$, then $\lambda(G) = \delta(G)$.

(b) Give an example of a simple graph $G$ with $\delta(G) = \lfloor |V(G)|/2 \rfloor - 1$ and $\lambda(G) < \delta(G)$.

*Proof.* (a)

*Proof from 例 4.3.1 in 图论及其应用 (徐俊明)(I don't know the corresponding index in the slide):* It is sufficient to prove that the diameter $d(G)$ of $G$ satisfies $d(G) \leq 2$. It is obvious, as every two non-adjacent vertices must have a common neighbor. Indeed, let $u$ and $v$ be non-adjacent, then

$$|N(u) \cap N(v)| \geq 2\delta(G) - (n-2) > 0$$

*Second proof:* We already know that $\lambda(G) \leq \delta(G)$(Elementary inequality in this course), so it is sufficient to prove that $\lambda(G) \geq \delta(G)$. For any edge cut $[X, \overline{X}]$, assume without loss of generality that $1 \leq |X| \leq \lfloor n/2 \rfloor$. Then,

$$\begin{aligned} \left|[X, \overline{X}]\right| &\geq \sum_{v \in X} \big(d(v) - |X| + 1\big) \\ &\geq |X|\big(\delta(G) - |X| + 1\big) \\ &= \delta(G) + (|X| - 1)(\delta(G) - |X|) \\ &\geq \delta(G), \end{aligned}$$

Thus, $\lambda(G) \geq \delta(G)$ by the definition of $\lambda(G)$.

(b) Consider the disjoint union of two equal-sized cliques, each of order at least two. This graph satisfies the degree condition, yet has an edge connectivity of 0. □

**2.** Let $G$ be a 3-connected graph with $|V(G)| \geq 5$. Prove that there exists $e \in E(G)$ such that $\kappa(G \cdot e) \geq 3$. (W. T. Tutte, 1961)

Here $G \cdot e$ denotes the simple graph obtained by contracting $e$ and then deleting loops and replacing parallel edges by single edges.

*Proof.* Suppose, for the sake of contradiction, that $G \cdot e$ is not 3-connected for any edge $e \in E(G)$. For an arbitrary edge $uv$, any vertex cut of size at most two in $G \cdot uv$ must contain the contracted vertex resulting from the contraction of $uv$; otherwise, such a cut would also separate $G$. Furthermore, this vertex cut cannot consist of the contracted vertex, since $G - \{u, v\}$ is connected. Consequently, there must exist a vertex $w$ distinct from $u$ and $v$ such that $G - \{u, v, w\}$ is disconnected.

Now, choose an edge $xy$, a corresponding vertex $z$ as described above, and a component $C$ of $G - \{x, y, z\}$ such that $|V(C)|$ is minimized among all such possible choices. Since $G$ is 3-connected, every component of $G - \{x, y, z\}$ must have an edge to each of $x, y, z$. Choose $v \in V(C)$ that is adjacent to $z$. There exists a vertex $w$ distinct from $z$ and $v$ such that $G - \{z, v, w\}$ is disconnected.

Finally, let $C'$ be a component of $G - \{z, v, w\}$ which contains neither $x$ nor $y$ ($x$ and $y$ must be contained in the same component if $w \notin \{x, y\}$). Then $V(C') \subseteq V(C) \setminus \{v\}$, as $C'$ must have an edge to $v$, which contradicts the minimality of $C$. □

**3.** Let $D$ be a digraph with $\kappa(D) \geq k$, where $k \geq 1$, and let $x_0, x_1, \ldots, x_k$ be distinct vertices of $D$. Prove that $D$ contains directed paths $P_i$ from $x_0$ to $x_i$, $1 \leq i \leq k$, which are pairwise vertex-disjoint outside $x_0$, and directed paths $Q_i$ from $x_i$ to $x_0$, $1 \leq i \leq k$, which are pairwise vertex-disjoint outside $x_0$.

*Proof.* Construct a new digraph $D'$ from $D$ by adding an auxiliary vertex $y$ along with the directed arcs $x_i y$ for $1 \leq i \leq k$. Consider an arbitrary vertex subset $S \subseteq V(D') \setminus \{x_0, y\}$ with $|S| < k$. At least one of $x_1, \ldots, x_k$ lies outside $S$, say $x_j$. Since $D - S$ remains strongly connected, it contains a directed path from $x_0$ to $x_j$. Appending $x_j y$ to this path yields a directed $x_0$-$y$ path in $D' - S$. Consequently, no set of fewer than $k$ vertices can separate $x_0$ from $y$ in $D'$.

By Menger's theorem, $D'$ contains $k$ internally vertex-disjoint directed $x_0$-$y$ paths. Since the in-degree of $y$ is exactly $k$, the terminal arcs of these paths must be distinct and are therefore precisely $x_1 y, x_2 y, \ldots, x_k y$. Deleting $y$ from these paths yields the desired paths $P_1, P_2, \ldots, P_k$ in $D$.

Finally, observe that reversing every arc of a digraph preserves its strong connectivity following the deletion of any set of fewer than $k$ vertices. By applying an analogous argument to the digraph obtained by reversing all arcs of $D$, and subsequently reversing the directed paths thus obtained, we deduce the existence of the desired paths $Q_1, Q_2, \ldots, Q_k$. □