Week 2

In the solutions, we may use some definitions that are frequently encountered in graph theory but have not yet been introduced in class. We list them here:

- **Independent set.** An *independent set* in a graph G is a set of vertices no two of which are adjacent.
- **Independence number $\alpha(G)$.** The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum size of an independent set in G.
- **Clique.** A *clique* in a graph G is a set of vertices that are pairwise adjacent. Equivalently, a clique is a complete subgraph of G.
- **F-free graph.** A graph G is said to be *F-free* if G contains no subgraph isomorphic to F.
- **Closed neighborhood $N[v]$.** For a vertex $v \in V(G)$, the *closed neighborhood* of v, denoted by $N[v]$, is
$$N[v] = N(v) \cup \{v\},$$
where $N(v)$ is the set of all neighbors of v.
- **Join of graphs.** The *join* of two vertex-disjoint graphs G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H by adding all edges between every vertex of G and every vertex of H.

In the provided solutions, we will use red to highlight key points that may be essential to the proofs.

Exercises

1. Prove that $R(3,4) = 9$.

Proof. (1) $R(3,4) \geq 9$. The following Wagner graph gives the required lower-bound construction:

The graph shown in Figure 1 (the Wagner graph) has order 8 and contains neither a clique of size three nor an independent set of size four. Indeed, the neighborhood of every vertex is an independent set, so the graph

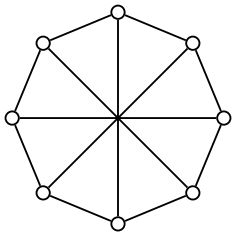

Figure 1: The Wagner graph.

is triangle-free. Moreover, any independent set of size four in the underlying 8-cycle would have to consist of four alternating vertices, but such vertices are joined by the additional edges of the Wagner graph. Hence, $\alpha(G) \leq 3$.

In general, graphs constructed to establish lower bounds for Ramsey numbers are called **Ramsey graphs**, and there is no general construction for such graphs.

(2) Upper bound: Let G be a graph of order 9. We prove that G contains either a clique of size 3 or an independent set of size 4. By the Handshaking Lemma, the sum of the degrees of all vertices is even, so G must contain a vertex of even degree. Let v be such a vertex.

Case 1. Suppose that $d(v) \geq 4$. Since

$$R(2,4) = 4,$$

the subgraph $G[N(v)]$ either contains an independent set of size 4, or contains a copy of K_2, which together with v forms a triangle.

Case 2. Suppose that $d(v) \leq 2$. Then $G - N[v]$ has at least 6 vertices. Since

$$R(3,3) = 6,$$

the graph $G - N[v]$ contains either a triangle or an independent set of size 3. In the latter case, these three vertices together with v form an independent set of size 4. □

Remark. For any integers $k, l \geq 2$, we have

$$R(k,l) \leq R(k-1,l) + R(k,l-1).$$

Moreover, if $R(k-1,l)$ and $R(k,l-1)$ are both even, then the inequality is strict; that is,

$$R(k,l) < R(k-1,l) + R(k,l-1).$$

2. Nine mathematicians meet at an international conference. Among any three of them, at least two speak the same language. Prove: if each mathematician speaks at most three languages, then at least three mathematicians can converse in the same language.
(7th US Mathematical Olympiad problem, May 2, 1978.)

First Proof. Suppose, to the contrary, that no three mathematicians can converse in the same language. Then every language is spoken by at most two of the nine mathematicians.

Construct a graph G whose vertices are the nine mathematicians, where two vertices are adjacent if the corresponding mathematicians speak a common language. Since each mathematician speaks at most three languages, and each language is spoken by at most one other mathematician, every vertex of G has degree at most 3. Hence

$$\Delta(G) \leq 3.$$

On the other hand, the hypothesis says that among any three mathematicians, at least two speak the same language. Therefore, among any three vertices of G, at least two are adjacent. Thus G has no independent set of size 3, i.e.,

$$\alpha(G) \leq 2.$$

Now let S be a maximal independent set of G. Since $\alpha(G) \leq 2$, we have

$$|S| \leq 2.$$

By the maximality of S, every vertex outside S has a neighbor in S. Therefore,

$$V(G) = \bigcup_{v \in S} N_G[v].$$

Since $\Delta(G) \leq 3$, for every $v \in V(G)$,

$$|N_G[v]| \leq 4.$$

Consequently,

$$9 = |V(G)| \leq \sum_{v \in S} |N_G[v]| \leq 4|S| \leq 8,$$

a contradiction. Hence our assumption was false. Therefore, some language is spoken by at least three mathematicians, and these three mathematicians can converse in the same language. □

Remark. Generally, we have

$$\alpha(G) \geq \frac{|V(G)|}{\Delta(G)+1}.$$

Second Proof. Recall from the first proof that $\Delta(G) \leq 3$ and $\alpha(G) \leq 2$. Since $\alpha(G) \leq 2$, the complement $\overline{G}$ is K_3-free. Hence, by Mantel theorem,

$$e(\overline{G}) \leq \left\lfloor \frac{9^2}{4} \right\rfloor = 20.$$

Therefore,

$$e(G) = \binom{9}{2} - e(\overline{G}) \geq 36 - 20 = 16.$$

On the other hand, since $\Delta(G) \leq 3$,

$$e(G) \leq \frac{1}{2}\Delta(G)|V(G)| \leq \frac{1}{2} \cdot 3 \cdot 9 = \frac{27}{2} < 16,$$

a contradiction.

□

3. Prove that $\mathrm{Aut}(C_n) = D_n$, where D_n denotes the dihedral group.

Proof. Let

$$V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\},$$

where the indices are taken modulo n.

Let $\varphi \in \mathrm{Aut}(C_n)$. Suppose

$$\varphi(v_0) = v_a.$$

Since v_1 is adjacent to v_0, its image must be one of the two neighbors of v_a. Hence either

$$\varphi(v_1) = v_{a+1}$$

or

$$\varphi(v_1) = v_{a-1}.$$

In the first case, adjacency determines the images of the remaining vertices successively, giving

$$\varphi(v_i) = v_{a+i} \qquad \text{for all } i.$$

Thus φ is a rotation.

In the second case, similarly,

$$\varphi(v_i) = v_{a-i} \qquad \text{for all } i,$$

so φ is a reflection.

Therefore every automorphism of C_n is either a rotation or a reflection. Conversely, every rotation and reflection of C_n preserves adjacency and hence is an automorphism.

Let

$$r(v_i) = v_{i+1} \qquad \text{and} \qquad s(v_i) = v_{-i}.$$

Then r is a rotation and s is a reflection, and

$$r^n = s^2 = 1, \qquad srs = r^{-1}.$$

Hence

$$\mathrm{Aut}(C_n) = \langle r, s \rangle = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1} \rangle \cong D_n.$$

□

4. Prove: Let G be a simple undirected graph with vertex set $V = \{x_1, x_2, \ldots, x_n\}$ containing no subgraph K_{r+1}. Then there exists a complete r-partite graph H with vertex set V such that for every i $(1 \le i \le n)$,

$$d_G(x_i) \le d_H(x_i),$$

with equality if and only if $G \cong H$. (P. Erdős, 1970)

Proof. We proceed by induction on r. For $r = 1$, the graph G contains no K_2, and hence G is an empty graph. Thus we may simply take $H = G$, and the assertion is immediate. Now let $r \ge 2$, and assume that the result holds for $r - 1$. Choose a vertex $v \in V(G)$ of maximum degree, and put

$$A = N_G(v), \qquad B = V(G) \setminus A.$$

Since G contains no K_{r+1}, the induced subgraph $G[A]$ contains no K_r. Indeed, a copy of K_r in $G[A]$, together with v, would form a copy of K_{r+1} in G. By the induction hypothesis, there exists a complete $(r-1)$-partite graph H_A with vertex set A such that

$$d_{G[A]}(x) \le d_{H_A}(x) \qquad \text{for every } x \in A.$$

Now construct a graph H on $V(G)$ as follows: keep H_A on A, make B an independent set, and join every vertex of A to every vertex of B. Clearly, H is a complete r-partite graph. We claim that

$$d_G(x) \le d_H(x) \qquad \text{for every } x \in V(G).$$

First, let $x \in B$. Since v has maximum degree,

$$d_G(x) \le |A| = d_H(x).$$

Next, let $x \in A$. We have

$$d_G(x) = d_{G[A]}(x) + |N_G(x) \cap B|.$$

Since $d_{G[A]}(x) \le d_{H_A}(x)$ and $|N_G(x) \cap B| \le |B|$, it follows that

$$d_G(x) \le d_{H_A}(x) + |B| = d_H(x).$$

Thus the required degree inequalities hold for every vertex.

It remains to consider the equality case. Suppose that $d_G(x) = d_H(x)$ for every $x \in V(G)$. For every $x \in A$, equality in

$$d_G(x) = d_{G[A]}(x) + |N_G(x) \cap B| \le d_{H_A}(x) + |B| = d_H(x)$$

implies $d_{G[A]}(x) = d_{H_A}(x)$ and $|N_G(x) \cap B| = |B|$. Hence every vertex of A is adjacent in G to every vertex of B. Moreover, by the induction hypothesis,

$$G[A] \cong H_A.$$

Now let $y \in B$. Since all edges between A and B are present, y already has all the $|A|$ vertices of A as neighbors. But

$$d_G(y) = d_H(y) = |A|,$$

so y has no neighbor in B. Thus B is an independent set in G. Consequently,

$$G = \overline{K}_{|B|} \vee G[A] \cong \overline{K}_{|B|} \vee H_A = H.$$

Therefore

$$d_G(x_i) \leq d_H(x_i) \qquad (1 \leq i \leq n),$$

with equality for every i if and only if $G \cong H$. □