Week 8

1. Let G be a graph of order n ($n \geq 4$), let $\Delta = \Delta(G)$, and let n_i denote the number of vertices of degree i in G. Prove that

(a) if G is a triangulation, then

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta + 12;$$

(b) if G is a tree, then

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2.$$

Proof. (a) Since G is a triangulation, we have $e(G) = 3n - 6$. Every vertex of a triangulation of order at least 4 has degree at least 3. Hence

$$\sum_{i=3}^{\Delta} in_i = \sum_{v \in V(G)} d_G(v) = 2e(G) = 6n - 12.$$

Rearranging gives the required identity.

(b) Since G is a tree, $e(G) = n - 1$ and every vertex has positive degree. Thus

$$\sum_{i=1}^{\Delta} in_i = 2e(G) = 2n - 2.$$

Rearranging gives

$$n_1 = 2 + \sum_{i=3}^{\Delta} (i-2)n_i,$$

which is the desired identity. □

2. Prove that

(a) if G is a connected planar graph of order n and girth $g \geq 3$, then

$$e(G) \leq \frac{g(n-2)}{g-2};$$

(b) the Petersen graph is non-planar.

Proof. (a) Fix a plane embedding of G. By the Handshaking Lemma and Euler's formula, we obtain

$$2e(G) = \sum_{f \in F(G)} d_G(f) \geq g|F(G)| = g(e(G) - n + 2).$$

Consequently, $(g-2)e(G) \leq g(n-2)$, as required.

(b) The Petersen graph has 10 vertices, 15 edges, and girth 5. If it were planar, (a) would imply

$$15 \leq \frac{5(10-2)}{5-2} = \frac{40}{3},$$

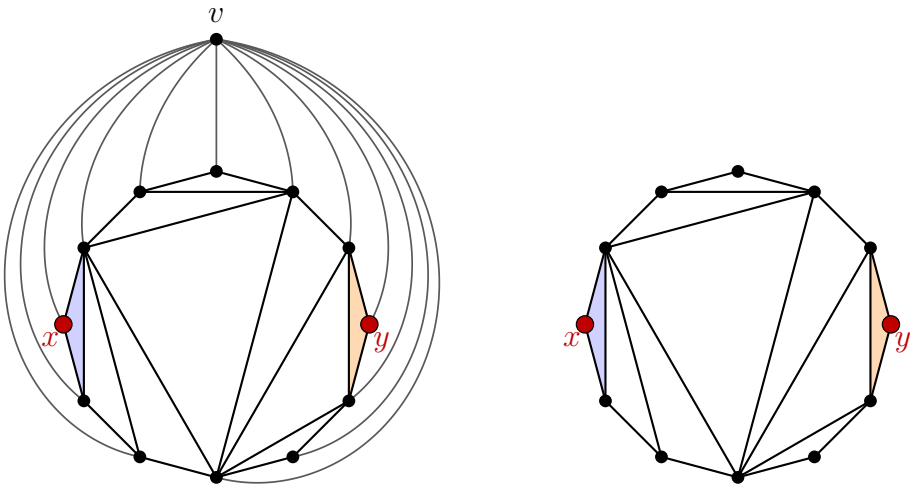
a contradiction. □

3. Let G be a simple planar graph with $\Delta(G) = n-1$. Prove that if $n \geq 5$, then G contains two nonadjacent vertices both of degree at most 3.

Proof. Choose a vertex v with $d_G(v) = n-1$. Add edges to G, without adding vertices, to obtain a maximal planar graph H. Choose a plane embedding of H with v on the boundary of the outer face. Since H is a triangulation, the neighbours of v, in their cyclic order around v, form a cycle C. Thus $H-v$ is a triangulated $(n-1)$-gon with outer boundary C (See the figure on the right).

Form an auxiliary graph T whose vertices are the triangles of this polygon, with two vertices adjacent when the corresponding triangles share a diagonal. The graph T is connected since a curve inside the polygon between points in two triangles, avoiding all polygon vertices, gives a walk between them. A triangulation of an $(n-1)$-gon has $n-3$ triangles and $n-4$ diagonals (see the second proof for details), so T is a tree with at least two vertices (the path between vertices on two sides of a diagonal must pass through this diagonal). It therefore has at least two leaves. Each leaf corresponds to a triangle with two polygon sides. The common endpoint of those two sides has degree 2 in $H-v$. Thus $H-v$ has two distinct vertices x, y of degree 2.

It follows that x and y are nonadjacent in H and satisfy $d_H(x) = d_H(y) = 3$. Since G is a subgraph of H, they are also nonadjacent in G and both have degree at most 3 in G.



□

Proof. Choose a vertex v with $d_G(v) = n-1$. Add edges to G, without adding vertices, to obtain a maximal planar graph H. Choose a plane embedding of H with v on the boundary of the outer face. Since H is a triangulation, the neighbours of v, in their cyclic order around v, form a cycle C. Thus $H-v$ is a triangulated $(n-1)$-gon with outer boundary C.

We use *discharging* to find two vertices of degree 2 in $H-v$.

Let f be the number of the inner faces in the triangulated polygon, and m the edges, by Euler's formula,

$$n-1-m+f = 1.$$

On the other hand, $3f + n - 1 = 2m$. Eliminating m, we have $f = n - 3$. Hence, $H - v$ has $n - 3$ bounded faces. Assign initial charge $n - 1$ to the outer face and initial charge -1 to each bounded face. The total charge is

$$(n - 1) - (n - 3) = 2.$$

Discharging rule: For each edge of C, transfer one unit of charge from the outer face to the bounded face incident with that edge.

After discharging, the outer face has charge 0. For each bounded face f, let $b(f)$ be the number of its boundary edges belonging to C. Its final charge is

$$\mathrm{ch}'(f) = b(f) - 1.$$

Since the polygon has at least four vertices, no bounded triangular face has all three edges in C. Hence $b(f) \leq 2$, and every bounded face has final charge at most 1. As the total charge remains 2, at least two bounded faces have final charge 1, and therefore each has two edges belonging to the polygon.

For each such face, the common endpoint of its two edges in C has degree 2 in $H - v$. Distinct such faces give distinct vertices, so there are two vertices x, y with

$$d_{H-v}(x) = d_{H-v}(y) = 2.$$

Thus x and y are nonadjacent in H and satisfy

$$d_H(x) = d_H(y) = 3.$$

Since G is a subgraph of H, they are also nonadjacent in G and both have degree at most 3 in G. □

4. Prove that

(a) if G is non-planar, then every subdivision of G is non-planar;

(b) if G is planar, then G contains no subdivision of K_5 or $K_{3,3}$;

(c) if G is planar, then every subgraph of G is planar.

Proof. (a) Suppose that a subdivision H of G has a plane drawing. For each edge of G, replace its corresponding path in H by an edge following the same curve. Since the subdivision paths are internally disjoint, this produces a plane drawing of G, a contradiction.

(b) If a planar graph contains a subdivision of either graph, deleting the other vertices and edges from a plane drawing gives a plane drawing of the subdivision of K_5 or $K_{3,3}$, contrary to (a) as K_5 and $K_{3,3}$ are non-planar.

(c) Given a plane drawing of G, delete the vertices and edges that do not belong to the chosen subgraph. The remaining drawing gives a plane drawing of the subgraph of G. □

5. Let G be a plane graph of odd order. Prove that if G has a Hamiltonian cycle, then G has an even number (≥ 2) of odd-degree faces.

Proof. Let C be a Hamiltonian cycle of G, and let n be the order of G. Write m_{in} and m_{out} for the numbers of edges drawn inside and outside C, respectively. Let F_{in} and F_{out} be the sets of faces inside and outside C, respectively.

Each edge of C contributes once to the sum of face degrees on either side, while every other edge contributes twice on its own side. Hence

$$\sum_{f \in F_{\text{in}}} d(f) = n + 2m_{\text{in}}, \qquad \sum_{f \in F_{\text{out}}} d(f) = n + 2m_{\text{out}}.$$

Both sums are odd as n is odd. Thus there is an odd, and therefore positive, number of odd-degree faces on each side of C. Their total number is consequently even and at least 2. □

6. A graph is called an *outerplanar graph* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains no subdivision of K_4 and $K_{2,3}$.

Proof. Let H be obtained from G by adding a new vertex v adjacent to every vertex of G. We first observe that

$$G \text{ is outerplanar} \iff H \text{ is planar.}$$

(**only if**) Taking subgraphs and suppressing subdivision vertices preserve outerplanarity. Neither K_4 nor $K_{2,3}$ is outerplanar. Thus an outerplanar graph contains neither of the forbidden subdivisions.

(**if**) Suppose that G is not outerplanar. Then H is non-planar. By Kuratowski's theorem, H contains a subdivision S of K_5 or $K_{3,3}$. We show that $S - v$, and hence G, contains a subdivision of K_4 or $K_{2,3}$.

If $v \notin V(S)$, then $S \subseteq G$ already contains a subdivision of K_4 or $K_{2,3}$, according as S is a subdivision of K_5 or $K_{3,3}$. Otherwise, omit one branch vertex v itself if it is a branch vertex, or an endpoint of the subdivision path containing v otherwise and retain the remaining branch vertices and their joining paths; these form a subdivision of K_4 or $K_{2,3}$, respectively, in G. □

7. Let G be a plane graph and let G^* be its geometric dual.

(a) Prove that G^* is a connected plane graph.

(b) Give an example showing that G need not be the geometric dual of G^*.

(c) Prove that if G is a connected plane graph, then

$$G^{**} \cong G.$$

Proof. (a) Place one dual vertex in each face of G. For every edge e of G, draw a dual edge e^* joining the vertices corresponding to its two incident faces, crossing e once and no other edge of G. When the two incident faces coincide, e^* is a loop. The arcs within each face can be chosen to meet only at the dual vertex, so the resulting dual drawing is plane.

To prove connectedness, choose two dual vertices and join them by a curve that avoids all vertices of G and meets its edges transversely at finitely many points. The sequence of faces visited by this curve gives a walk in G^* between the chosen vertices. Hence G^* is connected.

(b) Let G consist of two isolated vertices. It has one face and no edges, so $G^* = K_1$ and $G^{**} = K_1$. Thus $G^{**} \ncong G$, and G is not the geometric dual of G^*.

(c) If G has no edges, then $G = K_1$ and the assertion is immediate. Hence assume that G has at least one edge. Write $n = |V(G)|$, $m = e(G)$, and let f be the number of faces of G. Since G is connected, Euler's formula gives

$$n - m + f = 2.$$

The dual has f vertices and m edges, and it is connected by (a). Thus its number of faces is

$$m - f + 2 = n.$$

Every face of G^* contains a vertex of G. To see this, choose a dual edge e^* on its boundary. The corresponding primal edge e crosses e^* once and meets no other dual edge. Starting at the crossing and following e into the chosen face therefore leads to an endpoint of e within that face. There are n faces of G^* and n vertices of G, so each face of G^* contains exactly one vertex of G.

Use these original vertices as the dual vertices when forming G^{**}. For every edge $e = xy$ of G, the two faces of G^* incident with e^* contain x and y, respectively. Moreover, the original curve representing e crosses only e^*. Thus the original drawing of G is a geometric dual of G^*, and in particular $G^{**} \cong G$. □

8. Prove that no plane graph has five faces every two of which share a common boundary edge.

Proof. Suppose that a plane graph G has five such faces. In its geometric dual G^*, the five corresponding vertices are pairwise adjacent. Choosing one edge for each pair gives a subgraph isomorphic to K_5. But G^* is planar by Exercise 7(a), whereas K_5 is non-planar, a contradiction. □