Week 9

1. Prove that there is no convex polyhedron having an odd number of faces each of which has an odd number of edges.

Proof. Let F be the set of faces and m the number of edges. Since every edge lies on the boundary of exactly two faces,

$$\sum_{f \in F} d(f) = 2m.$$

Hence the number of faces having an odd number of edges must be even. □

2. Prove that every convex polyhedron has at least one triangular face or one vertex that is a trihedral angle.

Proof. Let n, m, and f denote the numbers of vertices, edges, and faces, respectively. A vertex is called trihedral if it has degree 3. Suppose there is no triangular face and no trihedral vertex. Then every face is bounded by at least four edges, and every vertex has degree at least four. Double counting gives

$$2m \geq 4f \qquad \text{and} \qquad 2m \geq 4n.$$

Hence $n + f \leq m$, which contradicts Euler's formula $n - m + f = 2$. □

3. Suppose that $N = (D_{xy}, \mathbf{c})$ is a capacity network and $B = (S, \overline{S})$ is an (x, y)-edge cut. Prove that for every (x, y)-flow $\mathbf{f}$ in N,

$$\mathbf{f}(B) - \mathbf{f}^-(S) = \mathbf{f}^+(S) - \mathbf{f}^-(S) = \begin{cases} \operatorname{val}\mathbf{f}, & \text{if } x \in S, \\ -\operatorname{val}\mathbf{f}, & \text{if } y \in S \text{(not follow course conventions).} \end{cases}$$

Proof. On summing the net outflow over all vertices of S, each edge with both ends in S is counted once positively and once negatively, and hence cancels. Consequently,

$$\mathbf{f}^+(S) - \mathbf{f}^-(S) = \sum_{v \in S} \big(\mathbf{f}^+(v) - \mathbf{f}^-(v)\big).$$

By the definition of a flow, every term in this sum is zero except possibly those at x and y. Moreover, since the net outflows over all vertices sum to zero, the net outflow at x is $\operatorname{val}\mathbf{f}$ and that at y is $-\operatorname{val}\mathbf{f}$. As S contains exactly one of x, y, the sum therefore equals $\operatorname{val}\mathbf{f}$, i.e.

$$\mathbf{f}^+(S) - \mathbf{f}^-(S) = \mathbf{f}^+(x) - \mathbf{f}^-(x) = \operatorname{val}\mathbf{f}.$$

Finally, since $B = (S, \overline{S})$, we have $\mathbf{f}(B) = \mathbf{f}^+(S)$. □

4. Let $N = (D_{xy}, \mathbf{c})$ be a capacity network, let $\mathbf{f}$ be an (x, y)-flow in N, and let $B = (S, \overline{S})$ be an (x, y)-edge cut. Prove that

(a) if $\mathbf{f}(a) = \mathbf{c}(a)$ for every $a \in (S, \overline{S})$ and $\mathbf{f}(a) = 0$ for every $a \in (\overline{S}, S)$, then $\mathbf{f}$ is a maximum flow and B is a minimum cut;

(b) if $\mathbf{f}$ is a maximum flow and B is a minimum cut, then $\mathbf{f}(a) = \mathbf{c}(a)$ for every $a \in (S, \overline{S})$ and $\mathbf{f}(a) = 0$ for every $a \in (\overline{S}, S)$.

Proof. By Exercise 3,

$$\mathbf{c}(B) - \operatorname{val}\mathbf{f} = \sum_{a \in (S,\overline{S})} \big(\mathbf{c}(a) - \mathbf{f}(a)\big) + \sum_{a \in (\overline{S},S)} \mathbf{f}(a) \geq 0. \tag{2}$$

Thus the value of every flow is at most the capacity of every (x, y)-cut.

(a) Under the given assumptions, both sums in (2) vanish, so $\operatorname{val}\mathbf{f} = \mathbf{c}(B)$. For every feasible flow $\mathbf{g}$ and every cut B' oriented from the source side,

$$\operatorname{val}\mathbf{g} \leq \mathbf{c}(B) = \operatorname{val}\mathbf{f} \leq \mathbf{c}(B').$$

Therefore $\mathbf{f}$ is a maximum flow and B is a minimum cut.

(b) For each edge $a = uv$, the residual network $D_{\mathbf{f}}$ has edges $a^+ : u \to v$ and $a^- : v \to u$ with residual capacities

$$c_{\mathbf{f}}(a^+) = \mathbf{c}(a) - \mathbf{f}(a), \qquad c_{\mathbf{f}}(a^-) = \mathbf{f}(a),$$

respectively, retaining only those with positive capacity.

If $D_{\mathbf{f}}$ contains an x–y path P, let

$$\varepsilon = \min_{b \in E(P)} c_{\mathbf{f}}(b) > 0.$$

Then

$$\mathbf{g}(a) = \begin{cases} \mathbf{f}(a) + \varepsilon, & a^+ \in E(P), \\ \mathbf{f}(a) - \varepsilon, & a^- \in E(P), \\ \mathbf{f}(a), & \text{otherwise} \end{cases}$$

defines a feasible flow with

$$\operatorname{val}\mathbf{g} = \operatorname{val}\mathbf{f} + \varepsilon > \operatorname{val}\mathbf{f},$$

contrary to the maximality of $\mathbf{f}$.

Let R be the set of vertices reachable from x in $D_{\mathbf{f}}$. Then $x \in R$, $y \notin R$, and no residual edge leaves R. Hence

$$\begin{aligned} a \in (R, \overline{R}) &\implies c_{\mathbf{f}}(a^+) = 0 \implies \mathbf{f}(a) = \mathbf{c}(a), \\ a \in (\overline{R}, R) &\implies c_{\mathbf{f}}(a^-) = 0 \implies \mathbf{f}(a) = 0. \end{aligned}$$

By Exercise 3,

$$\operatorname{val}\mathbf{f} = \sum_{a \in (R,\overline{R})} \mathbf{f}(a) - \sum_{a \in (\overline{R},R)} \mathbf{f}(a) = \mathbf{c}((R, \overline{R})).$$

Since B is a minimum cut, (2) gives

$$\operatorname{val}\mathbf{f} \leq \mathbf{c}(B) \leq \mathbf{c}((R, \overline{R})) = \operatorname{val}\mathbf{f}.$$

Thus

$$0 = \mathbf{c}(B) - \operatorname{val}\mathbf{f} = \sum_{a \in (S,\overline{S})} \underbrace{(\mathbf{c}(a) - \mathbf{f}(a))}_{\geq 0} + \sum_{a \in (\overline{S},S)} \underbrace{\mathbf{f}(a)}_{\geq 0}.$$

Therefore

$$\mathbf{f}(a) = \mathbf{c}(a) \quad \text{for every } a \in (S, \overline{S}), \qquad \mathbf{f}(a) = 0 \quad \text{for every } a \in (\overline{S}, S).$$

□