2026 Fall USTC Graph Theory Homework

Week11

Exercises

Exercise 1. Use the labelling method to find a maximum (x, y)-flow in the capacity network shown in Figure 9.

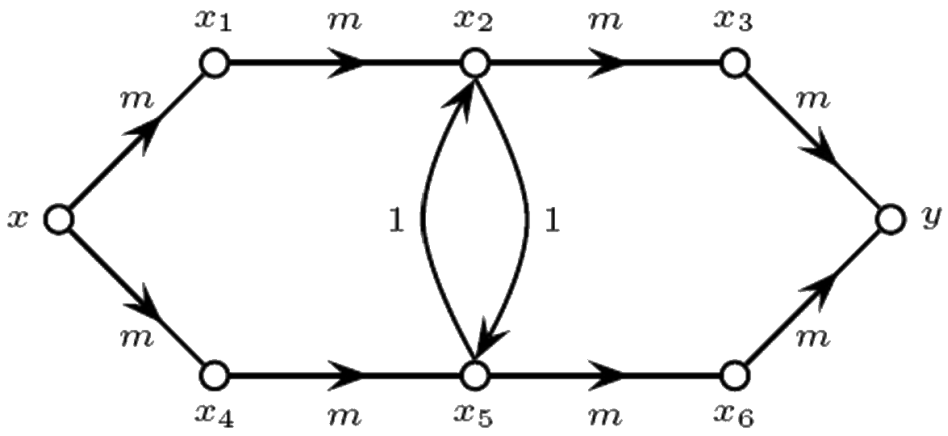

Figure 9: an application of the labelling method

Exercise 2. Prove that if $\mathbf{f}$ is a non-negative cycle vector of a digraph D, then

$$\mathbf{f} = \sum_{C \in \mathcal{C}} \lambda(C)\chi^C,$$

where $\lambda(C) \geq 0$, $\mathcal{C}(D)$ is the set of directed cycles of D, and χ^C is the incidence vector of the directed cycle C, i.e.

$$\chi^C(a) = \begin{cases} 1 & \text{if } a \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3. Let $\mathbf{f}$ be an (x, y)-flow, and let P be an $\mathbf{f}$ incrementing path of minimum cost $b(P, \mathbf{f})$ among all $\mathbf{f}$ incrementing paths; let $\widetilde{\mathbf{f}}$ be the revised flow based on P. (R. G. Busacker & P. J. Gowen, 1961)

(a) Prove that if $\mathbf{f}$ is a minimum-cost flow, then $\widetilde{\mathbf{f}}$ is also a minimum-cost flow.

(b) Prove that finding a minimum-cost $\mathbf{f}$ incrementing path in D is equivalent to finding a shortest xy-path in $D(\mathbf{f})$.