2026 Fall USTC Graph Theory Homework

Week6

Exercises

Exercise 1. Let T be a nontrivial tree. Prove:

(a) T is bipartite;

(b) Let $\{X, Y\}$ be the bipartition of T. Then:

(i) if $|X| \geq |Y|$, then X contains at least one vertex of degree 1;

(ii) if $|X| = |Y| + k$, then X contains at least $k+1$ vertices of degree 1.

Exercise 2. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of n distinct subsets of $X = \{1, 2, \ldots, n\}$. Prove that there exists $x \in X$ such that $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_n \cup \{x\}$ are all distinct.

Exercise 3. Prove: if B_1 and B_2 are bonds, then their symmetric difference $B_1 \triangle B_2$ is an edge-cut set and hence contains a bond.

Exercise 4. Let C be a cycle and B a bond of D. Prove:

(a) $\mathbf{f}_C \in \mathcal{E}(D)$ is a cycle vector of D, and all cycle vectors of D form a subspace of $\mathcal{E}(D)$;

(b) $\mathbf{g}_B \in \mathcal{E}(D)$ is a cut vector of D, and all cut vectors of D form a subspace of $\mathcal{E}(D)$.

Exercise 5. Let $\mathbf{K}$ be obtained from the incidence matrix $\mathbf{M}$ of a connected graph D by deleting any one row. Prove that $\mathbf{K}$ is a basis matrix of the cut space $\mathcal{B}(D)$.