2026 Fall USTC Graph Theory Homework

Week2[1]

Exercises

Exercise 1. Prove that $R(3,4) = 9$.

Exercise 2. Nine mathematicians meet at an international conference. Among any three of them, at least two speak the same language. Prove: if each mathematician speaks at most three languages, then at least three mathematicians can converse in the same language.
(7th US Mathematical Olympiad problem, May 2, 1978.)

Exercise 3. Prove that $\mathrm{Aut}(C_n) = D_n$, where D_n denotes the dihedral group.

Exercise 4. Prove: Let G be a simple undirected graph with vertex set $V = \{x_1, x_2, \ldots, x_n\}$ containing no subgraph K_{r+1}. Then there exists a complete r-partite graph H with vertex set V such that for every i $(1 \le i \le n)$,

$$d_G(x_i) \le d_H(x_i),$$

with equality if and only if $G \cong H$.
(P. Erdős, 1970)

Problems[2]

Problem 1. Determine the exact value of small Ramsey numbers:

- $R(5,5) = 43 \sim 46$: lower bound 43 (Exoo 1989); upper bound $48 \to 46$ (Angeltveit–McKay 2023, JGT 2026).
- $R(3,t)$: $R(3,10) = 40 \sim 41$, $R(3,11) = 47 \sim 50$.
- $R(4,t)$: $R(4,6) = 36 \sim 40$, $R(4,7) = 49 \sim 58$.

Problem 2. For any given cubic (3-regular) graph $G = (V,E)$, if there is a partition $X \cup Y$ of V such that

$$\Delta(X) \le 1, \qquad \Delta(Y) \le 1, \qquad \Big||X| - |Y|\Big| \le 2.$$

Problem 3. For any d-regular graph $G = (V,E)$, if there is a partition $X \cup Y$ of V such that

$$\delta(X) \ge \frac{d}{2}, \qquad \delta(Y) \ge \frac{d}{2}.$$

[1] The week number follows the academic calendar of the Academic Affairs Office; this course begins from the second week.

[2] Optional problems. Each Problem solved adds 10 points to the final grade.