2026 Fall USTC Graph Theory Homework

Week4&Week5[1]

Exercises

Exercise 1. Prove: a connected graph G is Eulerian if and only if every edge of G is contained in an odd number of cycles.

Exercise 2. Let G be a simple graph with no cut vertex and at least $n \geq 3$ vertices. Prove: if for any two vertices x and y at distance 2,

$$\max\{d_G(x),\, d_G(y)\} \geq \frac{n}{2} \qquad \text{(called the Fan condition)},$$

then G is Hamiltonian.
(see Geng-Hua Fan, *New sufficient conditions for cycles in graphs*, J. Combin. Theory Ser. B, **37** (1984), pp. 221–227.)

Exercise 3. Prove the following statement (Proposition 12.5): the line graph $L(G)$ of any graph G is claw-free.

Exercise 4. Let x, y, z be three distinct vertices of a strongly connected digraph D. Prove:

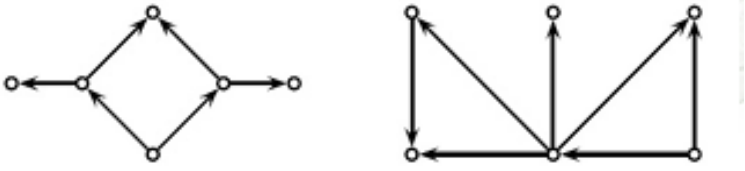
$$d_D(x, z) \leq d_D(x, y) + d_D(y, z).$$

Exercise 5. Let A be the adjacency matrix of a (directed or undirected) graph D, and I the $n \times n$ identity matrix. The polynomial

$$P_D(\lambda) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

is called the *characteristic polynomial* of D; the roots of $P_D(\lambda)$ are called the *characteristic roots* of D. The characteristic roots together with their multiplicities form the *spectrum*.

(a) Find the characteristic polynomials (both are of degree 6) and characteristic roots of the following two graphs:



[1]The week number follows the academic calendar of the Academic Affairs Office; this course begins from the second week.

Note: Graphs with the same spectrum as in the figure above are called *cospectral graphs*. Clearly, a graph uniquely determines its spectrum; the converse is one of the difficult open problems in graph theory.