2026 Fall USTC Graph Theory Homework

Week7

Exercises

Exercise 1. Let D be a loopless digraph, F a spanning forest of D, and $\mathbf{C}_F$ the basis matrices of the cycle space $\mathcal{C}(D)$ corresponding to F. Show that $\mathbf{C}_F$ is unimodular.

Exercise 2. Let D be a nontrivial connected loopless digraph and a any edge of D. Prove:

$$\tau(D) = \tau(D \cdot a) + \tau(D - a).$$

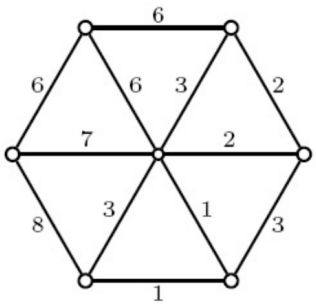
Here $D \cdot a$ is the digraph obtained from D by contracting a: delete a, identify its endpoints u and v as a single vertex, and retain all other edges, with their directions preserved.

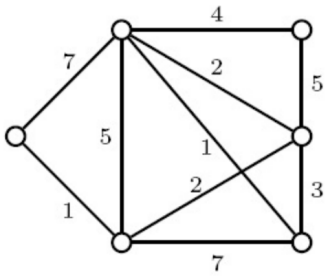
Exercise 3. The Kruskal algorithm for a minimum tree in a connected weighted simple graph (G, w) is as follows:

1. Choose $e_1 \in E(G)$ with $w(e_1)$ as small as possible;
2. once $e_1, e_2, \ldots, e_i$ are chosen, pick $e_{i+1} \in E(G) \setminus \{e_1, \ldots, e_i\}$ so that $G[\{e_1, \ldots, e_i, e_{i+1}\}]$ is acyclic and $w(e_{i+1})$ is as small as possible;
3. if $i < n - 1$, go to step 2; if $i = n - 1$, stop.

(a) Prove that the subgraph of G built by Kruskal's algorithm is a minimum tree of G.

(b) Use Kruskal's algorithm to find a minimum tree in the following graph.





Exercise 4. A company has branches in six cities $x_1, x_2, \ldots, x_6$. The direct airfare from x_i to x_j is the (i,j)-entry of the matrix below (∞ means no direct flight). Construct a table of the cheapest route between every pair of cities for the company.

$$\begin{pmatrix} 0 & 50 & \infty & 40 & 25 & 10 \\ 50 & 0 & 15 & 20 & \infty & 25 \\ \infty & 15 & 0 & 10 & 20 & \infty \\ 40 & 20 & 10 & 0 & 10 & 25 \\ 25 & \infty & 20 & 10 & 0 & 55 \\ 10 & 25 & \infty & 25 & 55 & 0 \end{pmatrix}$$

(Optional) **Exercise 5.** The following exercises concern the two conjectures stated in Section 7, and ask for a proof of the special cases in which they are known to hold. A *matching* of a graph $G = (V, E)$ is an edge subset in which no two edges are adjacent; a matching with exactly $\frac{|V(G)|}{2}$ edges is a *perfect matching* of G. A graph is *cubic* if every vertex has degree 3.

> **Conjecture (Berge–Fulkerson, 1971).** Every bridgeless cubic graph admits six perfect matchings such that every edge is covered exactly twice.
>
> **Conjecture (Berge, 1979).** Every bridgeless cubic graph admits five perfect matchings such that every edge belongs to at least one of them.

Prove the following special cases of the above conjectures:

(1) Both conjectures hold for cubic graphs that are 3-edge-colorable.

(2) Both conjectures hold for cubic Hamiltonian graphs (graphs containing a Hamilton cycle). A *Hamilton cycle* of G is a cycle containing every vertex of G.

(3) Do the above conjectures hold for cubic hypohamiltonian graphs? A graph G is *hypohamiltonian* if G itself is not Hamiltonian, but deleting any single vertex from G yields a Hamiltonian graph.