2026 Fall USTC Graph Theory Homework

Week8

Exercises

Exercise 1. Let G be a graph of order n ($n \geq 4$), let $\Delta = \Delta(G)$, and let n_i denote the number of vertices of degree i in G. Prove that

(a) if G is a triangulation, then

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta + 12;$$

(b) if G is a tree, then

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2.$$

Exercise 2. Prove that

(a) if G is a connected planar graph of order n and girth $g \geq 3$, then

$$e(G) \leq \frac{g(n-2)}{g-2};$$

(b) the Petersen graph is non-planar.

Exercise 3. Let G be a simple planar graph with $\Delta(G) = n - 1$. Prove that if $n \geq 5$, then G contains two nonadjacent vertices both of degree at most 3.

Exercise 4. Prove that

(a) if G is non-planar, then every subdivision of G is non-planar;

(b) if G is planar, then G contains no subdivision of K_5 or $K_{3,3}$;

(c) if G is planar, then every subgraph of G is planar.

Exercise 5. Let G be a plane graph of odd order. Prove that if G has a Hamiltonian cycle, then G has an even number (≥ 2) of odd-degree faces.

Exercise 6. A graph is called an *outerplanar graph* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains no subdivision of K_4 and $K_{2,3}$.

Exercise 7. Let G be a plane graph and let G^* be its geometric dual.

(a) Prove that G^* is a connected plane graph.

(b) Give an example showing that G need not be the geometric dual of G^*.

(c) Prove that if G is a connected plane graph, then

$$G^{**} \cong G.$$

Exercise 8. Prove that no plane graph has five faces every two of which share a common boundary edge.